Leggi Gamma

Funzione Gamma di Eulero

$$\Gamma(\alpha) = \int_0^{+\infty} \underline{x^{\alpha-1} e^{-x}}\, dx \quad , \alpha > 0$$

$\int$

Per $\alpha < 1$ $\quad x^{\alpha-1} \longrightarrow +\infty \quad$ per $x \to 0^+$

$x^{\alpha-1}$ è integrabile per $\alpha > 0$ nell'intorno di 0

e^{-x} è limitato nell'intorno di 0

$$\int_0^1 \frac{1}{x^{\beta}}\,dx < \infty \qquad \underline{\beta < 1} \qquad \beta = 1-\alpha < 1$$

$$\boxed{\alpha > 0}$$

$$\left.\frac{x^{-\beta+1}}{-\beta+1}\right|_0^1 =$$

$$\int_0^{\infty} \underbrace{x^{\alpha-1}}\,\underbrace{e^{-x}}\,dx$$

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} \, dx$$

$$\Gamma(1) = \int_0^{+\infty} e^{-x} \, dx = 1$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\int_0^{+\infty} \frac{1}{\sqrt{x}} e^{-x} \, dx$$

Proprietà

$$\Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha)$$

Dimostr.

$$\Gamma(\alpha+1) = \int_0^{+\infty} x^{\alpha} e^{-x}\, dx \overset{\text{per parti}}{=} \cancel{-e^{-x} \cdot x^{\alpha}}\Big|_0^{+\infty} -$$

$$- \int_0^{+\infty} \left(-e^{-x}\right) \alpha\, x^{\alpha-1}\, dx = \alpha \cdot \underbrace{\int_0^{+\infty} x^{\alpha-1} e^{-x}\, dx}_{=\Gamma(\alpha)}$$

$$= \alpha\,\Gamma(\alpha)$$

Applicazione

$$\Gamma(2) = \Gamma(1+1) = 1 \cdot \Gamma(1) = \underline{1}$$

$$\Gamma(3) = \Gamma(2+1) = 2 \cdot \Gamma(2) = 2 \cdot 1 = 2$$

$$\Gamma(4) = \Gamma(3+1) = 3\,\Gamma(3) = 3 \cdot 2 \cdot 1 = 6$$

$$\vdots$$

$$\Gamma(n) = (n-1)! \qquad \text{(per induzione)}$$

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

Densità : $\Gamma(\alpha, \lambda)$

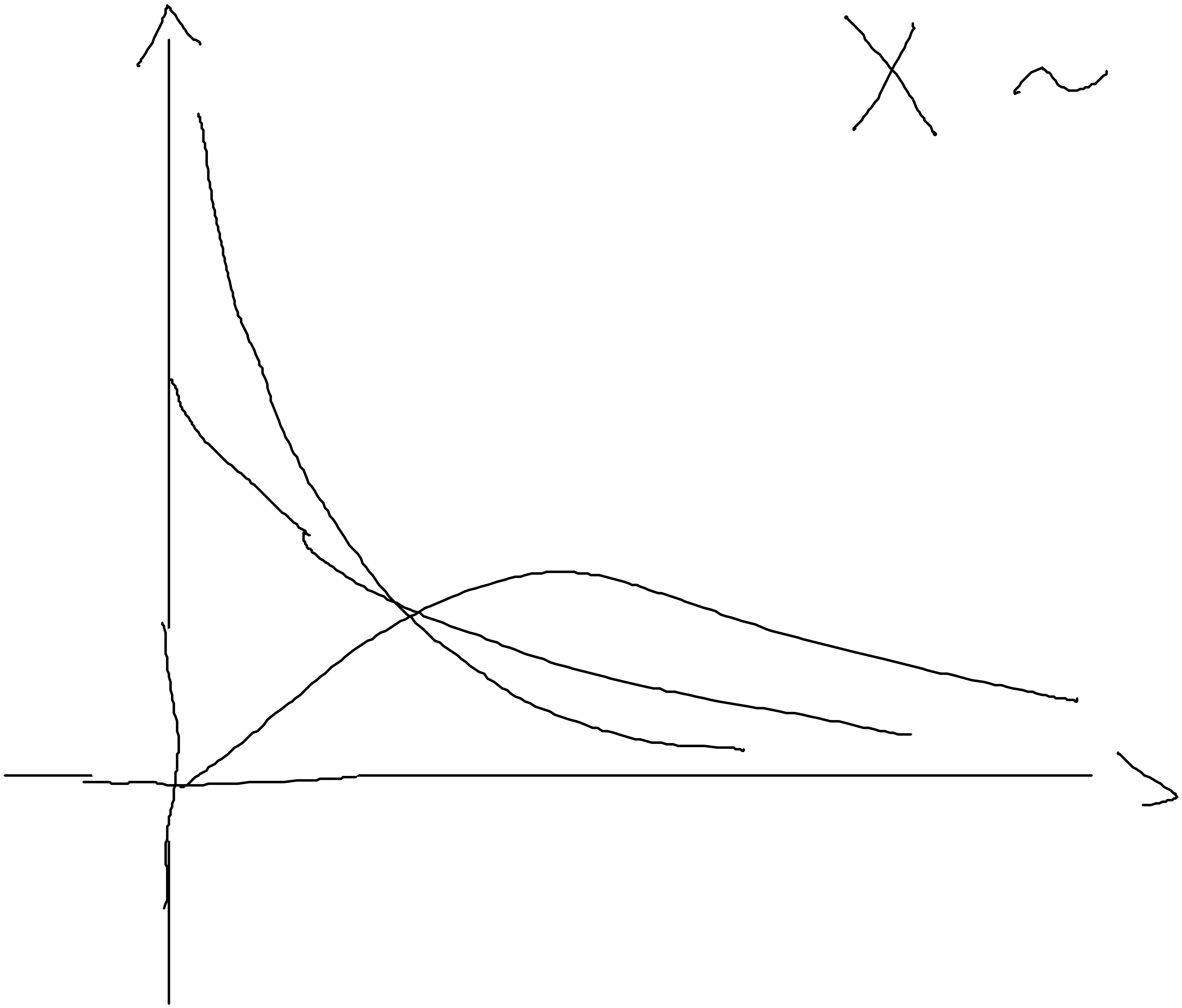
$$\begin{cases} \alpha > 0 \\ \lambda > 0 \end{cases}$$

$$\underline{f(x)} = \begin{cases} \dfrac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$\alpha = 1$

$$\begin{cases} \lambda e^{-\lambda x} \\ 0 \end{cases}$$

$X \sim f$



Sia X con densità $\Gamma(\alpha, \lambda)$

Calcoliamo $E[X^k]$ momento di ordine k

$$E[X^k] = \int_{-\infty}^{+\infty} x^k f(x)\, dx =$$

$$= \int_0^{+\infty} x^k \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\, dx$$

$$= \frac{\Gamma(\alpha+k)}{\lambda^{k+\alpha}} \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \underbrace{\int_0^{+\infty} \frac{\lambda^{k+\alpha}}{\Gamma(\alpha+k)} x^{(k+\alpha)-1} e^{-\lambda x} \, dx}_{=1} =$$

k intero

$$= \frac{\Gamma(\underline{\alpha+k})}{\lambda^k \cdot \Gamma(\alpha)} = \frac{(\alpha+k-1)\,\Gamma(\alpha+k-1)}{\lambda^k \Gamma(\alpha)} = \cdots$$

$K=2$

$$E[X^2] = \frac{\Gamma(\alpha+2)}{\lambda^2\,\Gamma(\alpha)} = \frac{(\alpha+1)\,\Gamma(\alpha+1)}{\lambda^2\,\Gamma(\alpha)} = \frac{(\alpha+1)\,\alpha\,\cancel{\Gamma(\alpha)}}{\lambda^2\,\cancel{\Gamma(\alpha)}} = \frac{\alpha(\alpha+1)}{\lambda^2}$$

$$\boxed{E[X] = \frac{\Gamma(\alpha+1)}{\lambda\,\Gamma(\alpha)} = \frac{\alpha\,\Gamma(\alpha)}{\lambda\,\Gamma(\alpha)} = \frac{\alpha}{\lambda}}$$

$$\mathrm{Var}\,X = E[X^2] - E^2[X] = \frac{\alpha(\alpha+1)}{\lambda^2} - \frac{\alpha^2}{\lambda^2} = \frac{\alpha}{\lambda^2}$$

In fact. for $\alpha=1$ $\mathrm{Var}\,X = \frac{1}{\lambda^2}$
(caso dell'esponenziale)
ustro...

Teorema. Siano X e Y 2 v. a. indipendenti con $X \sim \Gamma(\alpha_1, \lambda)$, $Y \sim \Gamma(\alpha_2, \lambda)$

Allora la v. a. $Z = X+Y \sim \Gamma(\alpha_1+\alpha_2, \lambda)$

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1) $$\left.\begin{array}{l} X \sim \Gamma(\alpha_1, \lambda) \\ Y \sim \Gamma(\alpha_2, \lambda) \\ Z \sim \Gamma(\alpha_3, \lambda) \end{array}\right\} \text{ indipendenti}$$

$$U = X+Y+Z \sim \Gamma(\alpha_1+\alpha_2+\alpha_3, \lambda)$$

$$U = \underbrace{(X+Y)}_{\Gamma(\alpha_1+\alpha_2, \lambda)} + \underbrace{Z}_{\sim \Gamma(\alpha_3, \lambda)}$$

$X+Y$ e Z sono indipendenti?

2) Senza l'indipendenza il risultato non vale

$$X \sim \Gamma(\alpha, \lambda)$$

$$Y = X \sim \Gamma(\alpha, \lambda)$$

se valesse il teorema, $X+Y$ dovrebbe avere legge $\underline{\Gamma(2\alpha, \lambda)}$

non è vero!

Densità normali (o Gaussiane)

Si chiama densità normale di parametri μ e σ^2 $(\mu \in \mathbb{R}, \ \sigma > 0)$ la densità definita da

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad x \in \mathbb{R}$$

$$\int_{-\infty}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \sqrt{2\pi}\,\sigma$$

Se $X \sim f$ si dice che

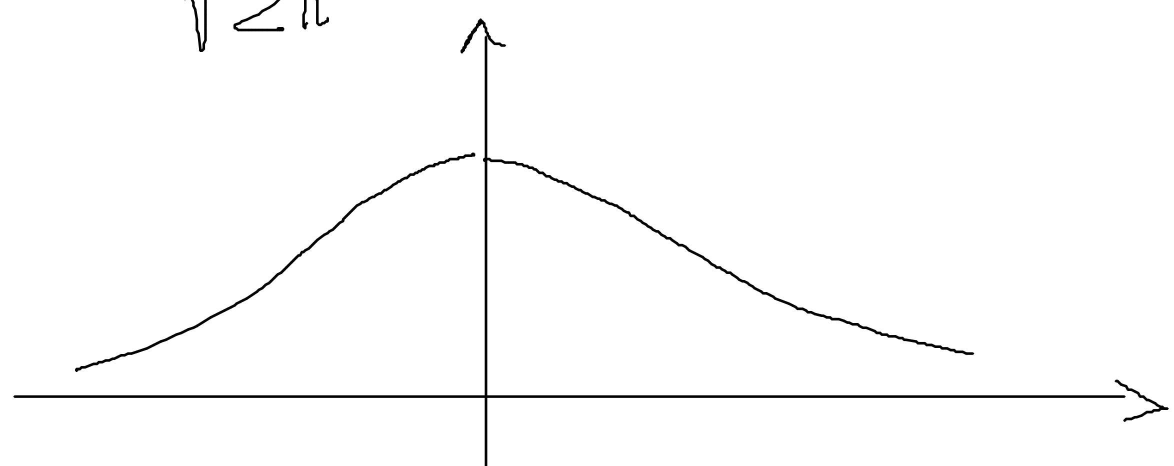
X ha densità normale di par. μ e σ^2

e si scrive $X \sim N(\mu, \sigma^2)$

La $N(0,1)$ si chiama densità normale standard

$\mu = 0$
$\sigma^2 = 1$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad x \in \mathbb{R}$$



$f(x) = f(-x)$

Calcoliamo $E[X]$ e $\mathrm{Var}\, X$

$$E[X] = \int_{-\infty}^{+\infty} x \, f(x) \, dx =$$

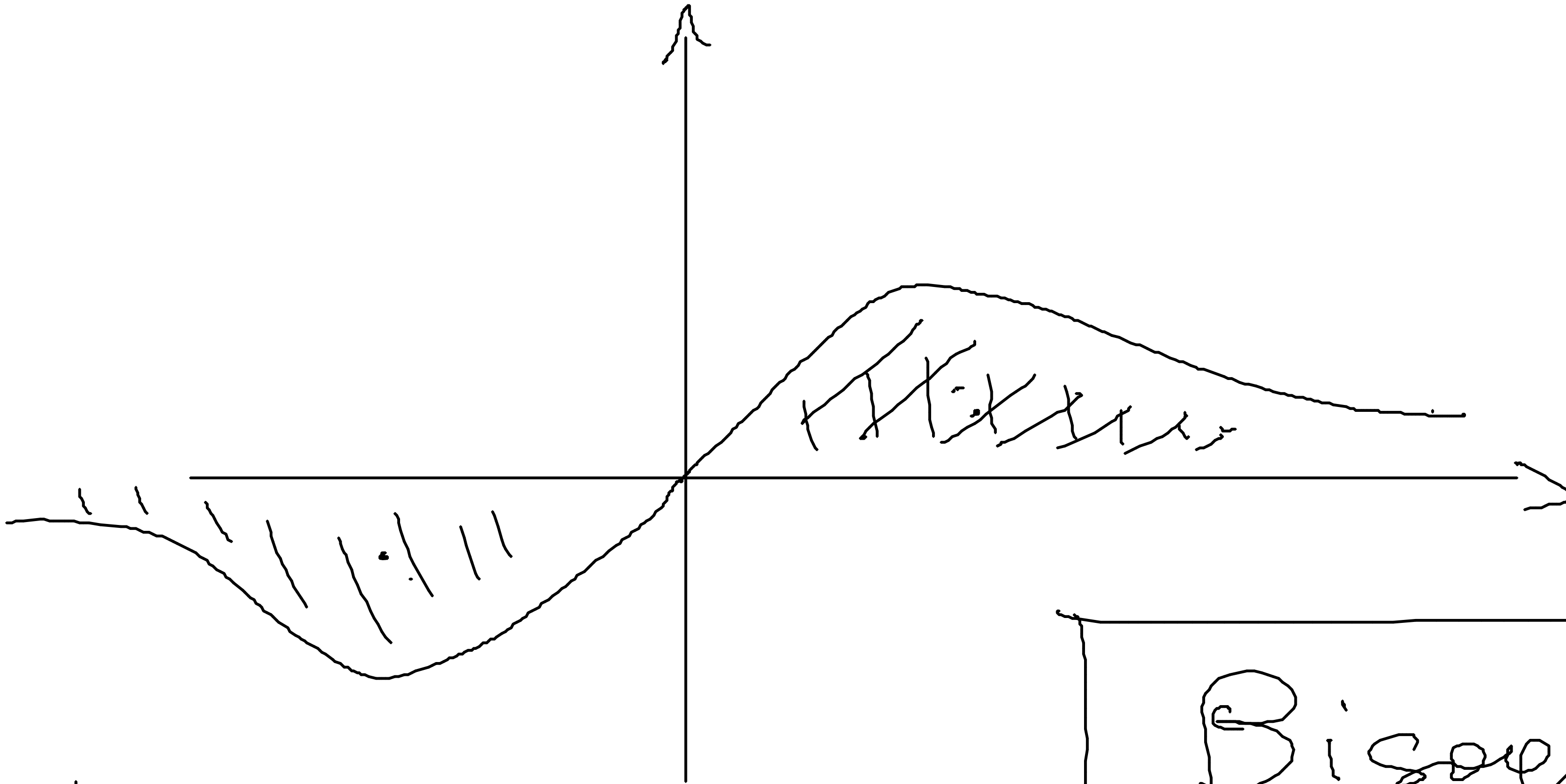
$$x e^{-x^2/2}$$

$$= \int_{-\infty}^{+\infty} x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[- e^{-\frac{x^2}{2}} \right]_{-\infty}^{+\infty} = 0$$

$$g(x) = x e^{-\frac{x^2}{2}}$$

$$g(-x) = -g(x)$$



$$\int_{-\infty}^{+\infty} g(x)\, dx = 0$$

Bisogna verificare

che $\int_0^{+\infty} g(x)\, dx < \infty$

$$\mathrm{Var}\, X = E[X^2] - E^2[X] =$$

$$= E[X^2] =$$

$$= \int_{-\infty}^{+\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^2 e^{-x^2/2} \, dx$$

$$(x) \cdot \left(x e^{-x^2/2} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \cancel{\left[-e^{-\frac{x^2}{2}} \cdot x \right]_{-\infty}^{+\infty}} - \int_{-\infty}^{+\infty} -e^{-x^2/2} \, dx \right] =$$

$$= \int_{-\infty}^{+\infty} \underbrace{\frac{1}{\sqrt{2\pi}} \, e^{-x^2/2}}_{} \, dx = 1$$

$$E[X] = 0$$

$$N(0, 1)$$

$$\operatorname{Var} X = 1$$

$$N(\mu, \sigma^2)$$

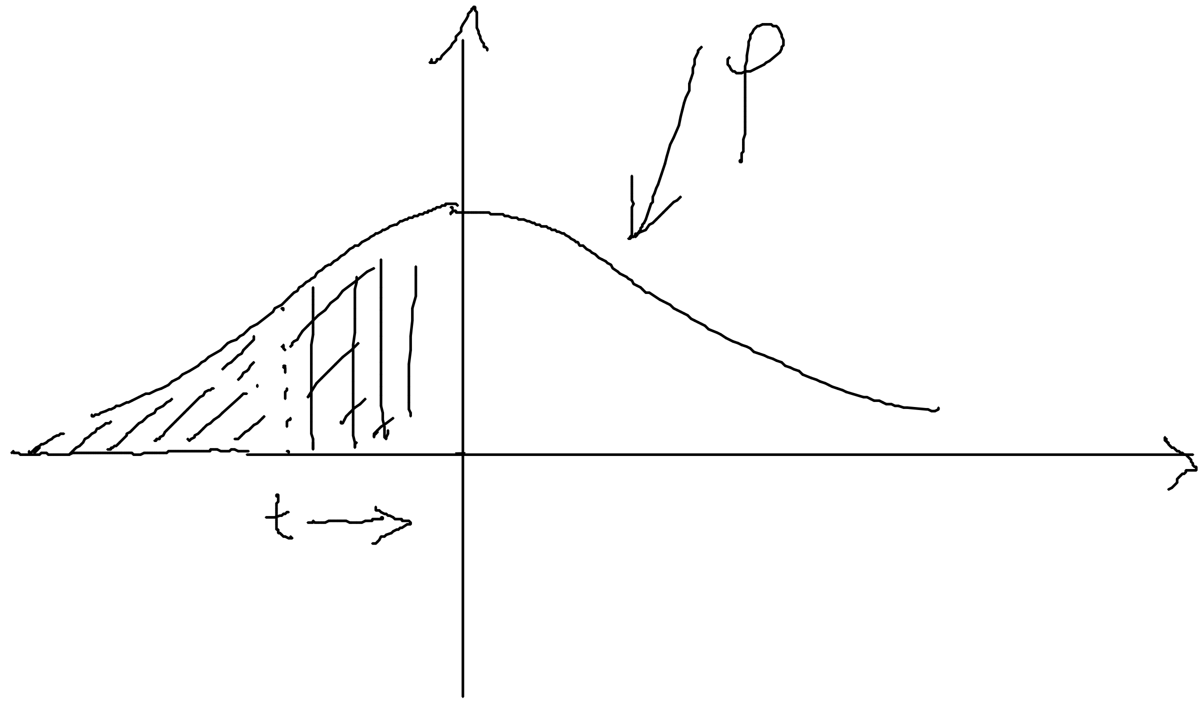
Funzione di ripartizione della $N(0,1)$

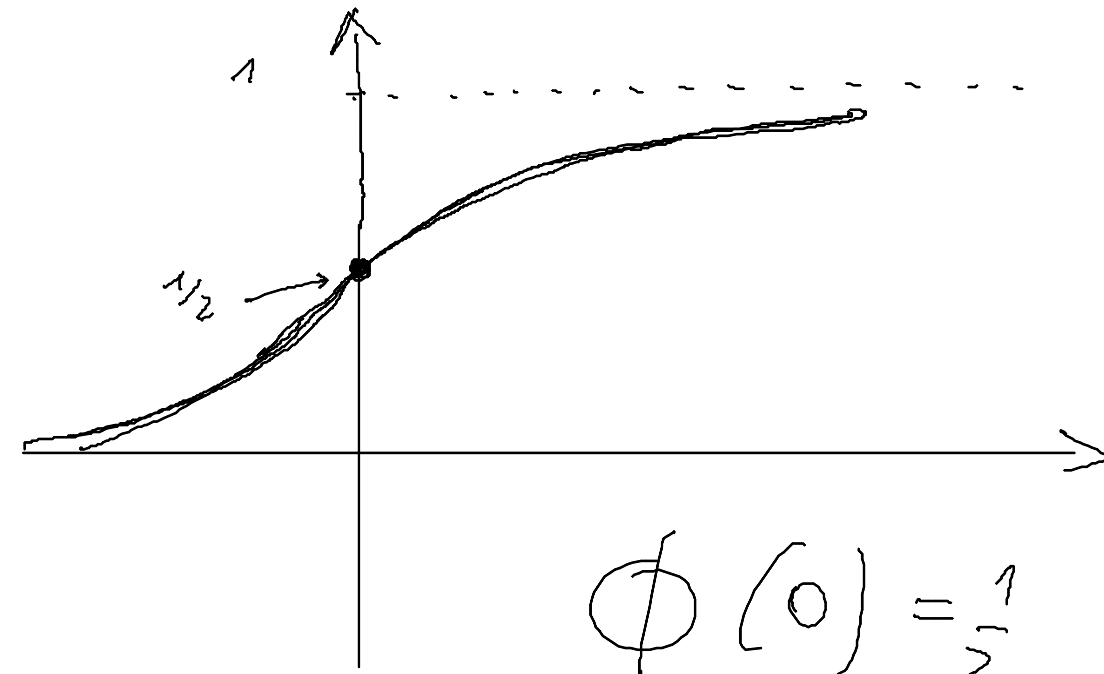
$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$\sim$ pratelli

Tavole statistich

$$\Phi(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$





$$\Phi(0) = \frac{1}{2}$$

$$\Phi(t) = \int_{-\infty}^{t} \varphi(x)\,dx$$

$$\Phi(t) + \Phi(-t) = 1$$

Definizione. Sia X una v. a.

Si dice che X è simmetrica se

X e $-X$ hanno la stessa legge.

cioè $\forall t$

$$P(X \leq t) = P(-X \leq t)$$

Supponiamo che X sia una v. a.
continua

$$P(X = t) = 0 \quad \forall t$$

$$\longrightarrow P(X \le t) = P(-X \le t) =$$

$$= P(X \ge -t) = 1 - P(X < -t)$$

$$= 1 - P(X \le -t)$$

$$P(X = -t)$$

Quindi, se X è continua e simmetrica, con f. d. r. F, la formula precedente diventa

$$P(X \le t) = 1 - P(X \le -t)$$

$$F(t) = 1 - F(-t)$$

$$F(t) + F(-t) = 1$$

$X \sim N(\mu, \sigma^2)$

<u>Proposizione</u> – Sia $X \sim N(\mu, \sigma^2)$ –

siano $a \neq 0$, $b \in \mathbb{R}$ due costanti.

Allora la v. a.

$$\underline{Y = aX + b} \sim N(a\mu + b, a^2\sigma^2)$$

Dimostraz. (i) $a > 0$

$$\underline{G(t) = P}(Y \le t) = P(aX + b \le t) =$$

$$= P(aX \le t - b) \overset{\downarrow}{=} P\left(X \le \frac{t-b}{a}\right) =$$

$$= F_X\left(\frac{t-b}{a}\right) = \underline{\underline{\int_{-\infty}^{\frac{t-b}{a}} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx}}$$

(ii) $a < 0$

$$G(t) = P\left(X \geq \frac{t-b}{a} \right) =$$

$$= 1 - P\left(X < \frac{t-b}{a} \right) =$$

$$= 1 - \int_{-\infty}^{\frac{t-b}{a}} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-} \text{----}$$

$$G'(t) = \frac{1}{\sqrt{2\pi}\,(a\sigma)}\, e^{-\frac{(t-(a\mu+b))^2}{2(a\sigma)^2}}$$

$$\frac{1}{\sqrt{2\pi}\;\underline{|a|\,\sigma}}\, e^{-\,\cdots}$$

$\boxed{\text{Corollario}}$. Siano $\mu \in \mathbb{R}$, $\sigma > 0$

(i) Se $X \sim N(0,1)$, allora

$$Y = \sigma X + \mu \sim N(\mu, \sigma^2)$$

(ii) Se $X \sim N(\mu, \sigma^2)$, allora

$$Y = \frac{X - \mu}{\sigma} \sim N(0,1)$$

Dim.

(ii) $\quad Y = \dfrac{X-\mu}{\sigma} = \underbrace{\left(\dfrac{1}{\sigma}\right)}_{a} X + \underbrace{\left(\dfrac{-\mu}{\sigma}\right)}_{b}$

$X \sim N(\mu, \sigma^2)$

$$\sim N\Big(\underbrace{\left(\tfrac{1}{\sigma}\right)\mu + \left(\tfrac{-\mu}{\sigma}\right)}_{a\mu+b}, \underbrace{\left(\tfrac{1}{\sigma}\right)^2 \sigma^2}_{a^2\sigma^2}\Big) = N(0, 1)$$

$X \sim N(\mu, \sigma^2)$.

Calcolare $E[X]$ e $\mathrm{Var}\, X$

$$Y = \frac{X-\mu}{\sigma} \sim \underline{\underline{N(0,1)}}$$

$$X = \sigma Y + \mu$$

$$E[X] = E[\sigma Y + \mu] = \sigma \underbrace{E[Y]}_{=0} + \mu = \mu$$

$$\mathrm{Var}\, X = \mathrm{Var}(\sigma Y + \mu) = \mathrm{Var}(\sigma Y) =$$

$$= \sigma^2 \underbrace{\mathrm{Var}\, Y}_{=1} = \sigma^2$$

$$\xi \to X \sim N(\mu, \sigma^2)$$

$$P(X \leq t) = P\left(\underbrace{\frac{X-\mu}{\sigma}}_{N(0,1)} \leq \frac{t-\mu}{\sigma} \right)$$

$$= \Phi\left(\frac{t-\mu}{\sigma} \right)$$

$$G'(t) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{\left(\frac{t-b}{a}-\mu\right)^2}{2\sigma^2}} \cdot \frac{1}{a} =$$

$$= \frac{1}{\sqrt{2\pi}\,(\sigma a)} e^{-\frac{\left(t-(a\mu+b)\right)^2}{2(a\sigma)^2}}$$

$$N\left(a\mu+b,\ a^2\sigma^2\right)$$

Sia ora $X \sim N(0,1)$

basta vedere che X è simmetrica

$$\underbrace{P(X \le t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx}$$

$$P(-X \le t) = P(X \ge -t) =$$

$f(x) = f(-x)$

$$= \int_{-t}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx =$$

$x = -y$

$dx = -dy$

$$= -\int_{t}^{-\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy =$$

$$= P(X \leq t)$$

$$\int_{-\infty}^{+\infty} x^n e^{-x^2} dx$$

$$\int x e^{-x^2} dx$$

$$\int e^{-x^2} dx$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_0^{+\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\,dx = \dots$$

$$\lambda x = y \longrightarrow x = \frac{y}{\lambda} \; ; \; dx = \frac{dy}{\lambda}$$

$$\dots = \int_0^{+\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{y^{\alpha-1}}{\lambda^{\alpha-1}} e^{-y} \frac{dy}{\lambda} = \frac{1}{\Gamma(\alpha)} \underbrace{\int_0^{+\infty} y^{\alpha-1} e^{-y}\,dy}_{=\Gamma(\alpha)}$$

$$= 1$$

Es. Sia $X \sim f$.

$$Z = X + Y = \underline{2X} \qquad \frac{d}{dt}\int_a^{t} f(x)\,dx = f(t)$$

Calcolare la legge di $2X$

$$G(t) = P(Z \leq t) = P(2X \leq t) =$$

$$= P\left(X \leq \frac{t}{2}\right) = \underline{\int_{-\infty}^{t/2} f(x)\,dx}$$

$$G'(t) = \frac{1}{2} f\left(\frac{t}{2}\right)$$

$$G'(t) = \frac{1}{2} f\left(\frac{t}{2}\right) = \begin{cases} \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \left(\frac{t}{2}\right)^{\alpha-1} \frac{1}{2} e^{-\lambda \frac{t}{2}} & \frac{t}{2} > 0 \\ 0 \qquad = \Gamma\left(\alpha, \frac{\lambda}{2}\right) & \frac{t}{2} \leq 0 \end{cases}$$

$$\frac{\left(\frac{\lambda}{2}\right)^{\alpha}}{\Gamma(\alpha)}$$

$$f(t) = \begin{cases} \frac{\mu^{\beta}}{\Gamma(\beta)} \boxed{t^{\beta-1}} e^{-\mu t} & t > 0 \\ 0 & t \leq 0 \end{cases} = \Gamma(\beta, \mu)$$

$$\frac{\lambda}{2} = \mu$$

$$\alpha = \beta$$